

New exactly solvable relativistic models with anomalous interaction¹

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Abstract

A special class of Dirac-Pauli equations with time-like vector potentials of external field is investigated. A new exactly solvable relativistic model describing anomalous interaction of a neutral Dirac fermion with a cylindrically symmetric external e.m. field is presented. The related external field is a superposition of the electric field generated by a charged infinite filament and the magnetic field generated by a straight line current. In non-relativistic approximation the considered model is reduced to the integrable Pron'ko-Stroganov model.

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1 Introduction

Exact solutions of relativistic wave equations are both very rare and important. Firstly they provide explicit solutions to concrete physical problems free of inaccuracies and inconveniences of the perturbation theory. Secondly, such exact solutions can serve as convenient basis sets for expanding the solutions of other physical problems which are not necessarily exactly solvable.

A good survey of exact solutions of relativistic wave equations can be found in [1]. Notwithstanding this book has been published as long ago as in 1990, it continues to be a good information resource on exactly solvable relativistic systems for particles with spins 0 and 1/2. Surely this collection is not exhausting: many new results have been obtained during two latest decades, including the problems for the Dirac equation in lower dimension space and the problems for neutral Dirac particles.

Exact solutions of the Dirac equation describing electrically neutral particles with non-minimal interaction with an external electromagnetic field are noteworthy. Physically, such solutions have a large applicative value since they can be used to model the motion of a neutron in realistic situations. In particular they have relations to the nuclear reactors security problems. Moreover, magnetic trapping of neutrons is a subject of direct experimental studies, refer, e.g., to [2]. Mathematically, the anomalous interaction terms depending on tensor fields present in such equations, dramatically reduce the number of problems which can be solved using complete separation of variables. In addition, just neutral particles anomalously coupled to external magnetic field give rise of the Aharonov-Casher effect [3] with its interesting physical and mathematical aspects.

The very possibility to exactly solve a quantum mechanical problem stems from the existence of a dynamical symmetry which is more extended than the geometric symmetry of the problem. The famous examples of such systems are the Kepler problems and isotropic oscillator whose dynamical symmetries are defined by groups $SO(4)$ and $U(3)$ respectively. One more well known example is the Pron'ko-Stroganov (PS) problem [4] which describes anomalous interaction of a non-relativistic electrically neutral particle of spin 1/2 with the field of straight line current. The related dynamical symmetry of negative energy states is described by group $SO(3)$ while the geometrical symmetry of the system is reduced to the rotation group in two dimensions, i.e., to $SO(2)$. Let us stress that the formulation of the PS problem uses the Schrödinger-Pauli equation for neutral particles, i.e., it is essentially nonrelativistic.

Paper [4] was followed by a number of publications devoted to exactly solvable problems for neutral particles. In particular, the supersymmetric aspects of the PS model were investigated in [5]–[7], more realistic models based on magnetic field produced by current of thin filament were discussed in [8] and [6] following an approach non-relativistic. A rather completed study of the Dirac-Pauli equation for neutral particles can be found in [9], the case of purely electric time-independent external field was studied in [10]. However, an exactly solvable relativistic analogue of the PS

problem was not known till now.

We can add that searching for exact solutions of Dirac equation belongs to evergreen problems, apparently the most recent result in this field can be found in [11]. Exactly solvable two-particle Dirac equations are discussed, e.g., in [12]. For exact solutions of relativistic wave equations for particles with higher and arbitrary spins see [13] and [14].

In the present paper we discuss a certain class of relativistic problems describing anomalous interaction of the Dirac fermion with an external electromagnetic field. The considered equations admit an effective reduction to equations invariant with respect to the 1+2 dimensional Galilei group. We show that this class includes new exactly solvable relativistic systems and we study in detail one of them, namely the one closely related to both the Pron'ko-Stroganov problem and relativistic problems for the neutron interacting with an external field. The corresponding external field is a superposition of the magnetic field generated by a straight line constant current and the electric field of a charged infinite filament. We will show that this system can be treated as an exactly solvable relativistic analogue of the Pron'ko-Stroganov problem.

In spite of that our main goal is to present an exactly solvable problem for neutral fermions, for the sake of generality we consider also more general problems with both the minimal and anomalous interactions. We also indicate an exactly solvable model of this general type, see Section 7.

2 Dirac-Pauli equations and reduction $\mathbf{SO}(1,3) \rightarrow \mathbf{HG}(1,2)$

Consider the Dirac-Pauli equation for a charged particle which interacts anomalously with an external electromagnetic field:

$$(\gamma^\mu \pi_\mu - m - \lambda S^{\mu\nu} F_{\mu\nu}) \psi = 0. \quad (1)$$

Here $\pi_\mu = p_\mu - eA_\mu$, $p_\mu = i\frac{\partial}{\partial x^\mu}$, A_μ being components of the vector-potential of the external electromagnetic field, γ^μ are Dirac matrices satisfying the Clifford algebra

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \quad (2)$$

$g^{\mu\nu}$ is the metric tensor whose nonzero elements are $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$, $S^{\mu\nu} = i(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)/4$ is the spin tensor, $F_{\mu\nu} = i[\pi_\mu, \pi_\nu]$ is the tensor of the electromagnetic field such that $F_{0a} = -E_a$, $F_{ab} = \epsilon_{abc} B_c$ where E_a and B_c are components of vectors of the electric and magnetic field strengths. In addition, e and λ denote particle charge and constant of anomalous coupling. The latest is usually represented as

$$\lambda = g\mu_0 \quad (3)$$

where μ_0 is the Bohr magneton and g is the Lande factor.

We use Heaviside units with $\hbar = c = 1$.

Equation (1) describes both the minimal and anomalous interaction of a Dirac fermion with an external electromagnetic field. Setting in (1) $\lambda = 0$ and supposing $e \neq 0$ we come to an equation describing the anomalous interaction only, while for $e = 0$, $\lambda \neq 0$ we have a Dirac-Pauli equation describing a neutral fermion.

Equation (1) is transparently invariant with respect to the Lorentz group $\text{SO}(1,3)$ which transforms time and space variables x_0, x_1, x_2, x_3 between themselves. Among the subgroups of this group there is the homogeneous Galilei group $\text{HG}(1,2)$ which includes the transformations of variables τ, x_1, x_2 where $\tau = (x_0 + x_3)/2$ (for all non-equivalent subgroups of $\text{SO}(1,3)$ and of the Poincaré group see [15]).

To search for exactly solvable problems based on equations (1) we restrict ourselves to special class of external fields, which makes it possible to expand solutions of (1) via solutions of reduced equations invariant w.r.t. group $\text{GH}(1,2)$. In other words, we will discuss such external fields for which these reduced equations be integrable.

To this end we first suppose that the vector-potential $A = (A_0, A_1, A_2, A_3)$ be light-like, i.e.,

$$A_\mu A^\mu = 0. \quad (4)$$

This condition can be always satisfied up to gauge transformations and so it does not lead to any loss of generality. Then we restrict ourselves to the vector-potentials of the following special form compatible with (4):

$$A = (\varphi, 0, 0, \varphi) \quad (5)$$

where φ is a function of time and spatial variables. In addition we suppose that φ depends on three variables only, namely,

$$\varphi = \varphi(\tau, x_1, x_2). \quad (6)$$

Vector-potentials (5), (6) satisfy the Lorentz gauge condition $p_\mu A^\mu = 0$ identically and the invariants of the related external field are both equal to zero, i.e.,

$$F_{\mu\nu} F^{\mu\nu} = 0 \text{ and } \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} = 0. \quad (7)$$

For convenience we fix a nonstandard realization of the Dirac matrices and set

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} \mathbf{0} & I \\ -I & \mathbf{0} \end{pmatrix}, \quad \gamma^\alpha = i \begin{pmatrix} -\sigma_\alpha & \mathbf{0} \\ \mathbf{0} & \sigma_\alpha \end{pmatrix} \quad (8)$$

where $\alpha = 1, 2$, σ_α are Pauli matrices, $\mathbf{0}$ and I are the 2×2 zero and unit matrix correspondingly. Then

$$S^{0\alpha} = \frac{1}{2} \begin{pmatrix} \mathbf{0} & \sigma_\alpha \\ -\sigma_\alpha & \mathbf{0} \end{pmatrix}, \quad S^{3\alpha} = \frac{1}{2} \begin{pmatrix} \mathbf{0} & \sigma_\alpha \\ \sigma_\alpha & \mathbf{0} \end{pmatrix}. \quad (9)$$

System (1) with a particular class of the vector-potential given by equations (5) and (6) is homogeneous with respect to the difference of independent variables $x_0 - x_3$. Thus it is convenient to rewrite it in the light cone variables

$$\tau = \frac{1}{2}(x_0 + x_3) \text{ and } \xi = x_0 - x_3.$$

As a result we obtain:

$$L\psi \equiv (\tilde{\gamma}_\mu \tilde{\pi}^\mu - m - \lambda \eta_\alpha F_\alpha) \psi = 0 \quad (10)$$

where $F_\alpha = \frac{\partial \varphi}{\partial x_\alpha}$, $\alpha = 1, 2$,

$$\begin{aligned} \tilde{\gamma}_0 &= \gamma_0 + \gamma_3, \quad \tilde{\gamma}_3 = \frac{1}{2}(\gamma_0 - \gamma_3), \quad \tilde{\gamma}_\alpha = \gamma_\alpha, \quad \eta_\alpha = \frac{1}{2}(\gamma_0 \gamma_\alpha + \gamma_3 \gamma_\alpha), \\ \tilde{\pi}_0 &= i \frac{\partial}{\partial \tau} - 2e\varphi, \quad \tilde{\pi}_3 = P_3 = -i \frac{\partial}{\partial \xi}, \quad \tilde{\pi}_\alpha = p_\alpha = -i \frac{\partial}{\partial x_\alpha} \end{aligned} \quad (11)$$

and summation w.r.t. repeated indices μ and α is imposed over the values $\mu = 0, 1, 2, 3$ and $\alpha = 1, 2$ respectively. In addition, we impose on solutions of (10) the standard condition of square integrability and ask for $\psi \rightarrow 0$ when $x_\alpha \rightarrow 0$.

Operator P_3 commutes with L and so is a constant of the motion for equation (10). Let us expand solutions of this equation via eigenvectors ψ_M of P_3 :

$$P_3 \psi_M = M \psi_M \Rightarrow \psi_M = \exp(iM\xi) \psi(\tau, x_1, x_2). \quad (12)$$

Let us denote

$$\psi(\tau, x_1, x_2) = \begin{pmatrix} \rho(\tau, x_1, x_2) \\ \chi(\tau, x_1, x_2) \end{pmatrix} \quad (13)$$

where ρ and χ are two-component spinors. Substituting (12) and (13) into (10) and using realization (8) of γ -matrices we obtain the following system:

$$(i\sigma_\alpha p_\alpha - m) \rho + \left(i \frac{\partial}{\partial \tau} - 2e\varphi - \lambda \sigma_\alpha F_\alpha \right) \chi = 0, \quad (14)$$

$$2M\rho - (i\sigma_\alpha p_\alpha + m)\chi = 0. \quad (15)$$

It is easy to convince oneself from eqs.(14), (15) that, without loss of generality we can assume that M cannot take the zero value. Indeed, setting $M = 0$ in (15) we reduce it to the equation for χ which does not have non-trivial normalizable solutions. Then, equating χ to zero in (14), we obtain the equation for ρ whose normalizable solutions are trivial also.

It is interesting to note that this system is nothing but a (1+2)-dimensional version of the Galilei-invariant Lévi-Leblond equation [16] with anomalous interaction, as can be immediately deduced by comparing with equation (52) for $e = k = 0$ in [17].

Galilei invariance of system (14), (15) can be easily verified since the Galilei boost for coordinates x_α :

$$x_\alpha \rightarrow x'_\alpha = x_\alpha + v_\alpha \tau, \quad \tau \rightarrow \tau' = \tau$$

where v_α are the boost parameters, keeps this system invariant provided F_a and ψ_M cotransform as

$$F_a \rightarrow F_a, \quad \psi_M \rightarrow (1 + \frac{1}{2} \eta_a v_a) \exp \left(M(v_\alpha x_\alpha + \frac{\tau v_\alpha v_\alpha}{2} + C) \right) \psi_M \quad (16)$$

where C is an arbitrary constant. Transformations (16) belong to a representation of the Galilei group in (1+2)-dimensional space.

Thus the Dirac-Pauli equation (1) can be reduced to the set of Galilei invariant equations (14), (15) provided the related vector-potential of the external electromagnetic field belongs to the class described by formulae (5) and (6). Solving eq.(15) for ρ under the condition $M \neq 0$ and substituting it into eq.(14) we obtain the Schrödinger-Pauli equation for the two-component spinor χ :

$$i \frac{\partial \chi}{\partial \tau} = \left(\varepsilon_0 + \frac{p^2}{2M} + 2e\varphi + \lambda \sigma_\alpha F_\alpha \right) \chi \quad (17)$$

where $\varepsilon_0 = \frac{m^2}{2M}$ and $p^2 = p_1^2 + p_2^2$.

Surely the non-relativistic equations (17) are more easy to handle then the initial relativistic equation (1), since they include smaller numbers of dependent and independent variables. In particular, a number of exactly (and quasi-exactly) solvable Schrödinger-Pauli equations (17) is well studied, and many of them can be used to construct solvable relativistic problems using the scheme inverse to the previously proposed.

In Section 4 we use this idea to generate a relativistic analogue of the PS problem.

3 Cylindrically symmetric potentials

Consider in more details a physically interesting subclass of equations (1), (5), (6) when the corresponding potential φ depends on the square $x^2 = x_1^2 + x_2^2$ of 2-vector $\mathbf{x} = (x_1, x_2)$ and is independent on τ . The related reduced equation (17) takes the form

$$i \frac{\partial \chi}{\partial \tau} = H \chi \quad (18)$$

where

$$H = \varepsilon_0 + \frac{p^2}{2M} + 2e\varphi + \lambda \frac{\sigma_\alpha x_\alpha}{x} \frac{\partial \varphi}{\partial x}. \quad (19)$$

Equation (18) has three additional constants of motion, namely,

$$P_0 = i \frac{\partial}{\partial \tau}, \quad J_{12} = x_1 p_2 - x_2 p_1 + \frac{i}{2} \sigma_3 \quad \text{and} \quad Q = \sigma_1 R_1 \quad (20)$$

where R_1 is the reflection operator which acts on χ as follows:

$$R_1 \chi(\tau, x_1, x_2) = \chi(\tau, -x_1, x_2).$$

Operators P_0 and J_{12} are generator of shifts w.r.t. variable τ and rotation generator correspondingly. They commute with the Hamiltonian (19) and between themselves. Expanding solutions of (18) via complete sets of eigenfunctions of P_0 and J_{12} it is possible to separate variables in this equation.

Operator Q represents a discrete symmetry w.r.t. the reflection of the first coordinate axis. It commutes with P_0 and H (19) but *anticommutes* with J_{12} . It follows from the above that eigenvalues of P_0 and H should be degenerated w.r.t. the sign of eigenvalues of J_{12} .

Notice that equation (18) admits other discrete symmetries like reflections of x_2 or both x_1 and x_2 . But all such additional symmetries are either rotation transformations or products of reflection Q and rotations.

Let us separate variables in equation (18). First we define the eigenvectors of P_0 which have the following form:

$$\chi_\varepsilon = \exp(-i\varepsilon\tau)\chi(\mathbf{x}). \quad (21)$$

Then, substituting eq.(21) into eq.(18) we obtain the equation

$$\varepsilon\chi = H\chi \quad (22)$$

where H is Hamiltonian (19).

In addition to the coupling constants e and λ , equation(22) includes two parameters, i.e., ε and M . For a fixed non-zero M this equation defines an eigenvalue problem for ε .

Now we can use the symmetry of (22) w.r.t. the rotation group (whose generator is J_{12}) to separate radial and angular variables. To do this we rewrite equation (22) in terms of angular variables, i.e., set $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $r = 2M\tilde{\lambda}x$ (where $\tilde{\lambda}$ is a normalizing parameter), and expand χ via eigenfunctions of the angular momentum operator J_{12} :

$$\chi = C_k \chi_k, \quad \chi_k = \frac{1}{\sqrt{r}} \begin{pmatrix} \exp(i(k - \frac{1}{2})\theta)\phi_1 \\ \exp(i(k + \frac{1}{2})\theta)\phi_2 \end{pmatrix} \quad (23)$$

where C_k are constants, ϕ_1 and ϕ_2 are functions of r and summation is imposed over the repeated indices $k = 0, \pm 1, \pm 2, \dots$.

In the following we restrict ourselves to solutions χ_k which correspond to non-negative values of k . Then solutions with k negative would be obtained by acting on χ_k by operator Q (20).

Substituting eq.(23) into eq.(22) we come to the following system:

$$H_k \phi \equiv \left(-\frac{\partial^2}{\partial r^2} + k(k - \sigma_3) \frac{1}{r^2} + 2e\varphi + \sigma_1 \frac{\lambda}{\tilde{\lambda}} \frac{\partial \varphi}{\partial r} \right) \phi = \tilde{\varepsilon} \phi \quad (24)$$

with $\phi = \text{column}(\phi_1, \phi_2)$, and

$$\tilde{\varepsilon} = (\varepsilon - \varepsilon_0)/2M\tilde{\lambda}^2. \quad (25)$$

Thus we reduce (22) to the system of two ordinary differential equations for radial functions, given by formula (24). Its solutions must be normalizable and vanish at $r = 0$. For some types of potential φ (and particular restrictions imposed on the coupling constants e and λ) this system is integrable and its solutions can be expressed via special functions. In the following section we consider an example of integrable equation (24) which corresponds to a neutral particle interacting anomalously with an external field.

4 Relativistic analog of PS problem

Let us set $e = 0$ in (14), (15) and choose the following particular realization for the potential φ :

$$\varphi = \omega \log(x) \quad (26)$$

where ω is a constant. Then the related equations (18) and (24) are reduced to the following forms

$$\varepsilon' \chi = \left(\frac{p^2}{2M} + \tilde{\lambda} \frac{\sigma_\alpha x_\alpha}{x^2} \right) \chi, \quad \varepsilon' = \varepsilon - \varepsilon_0 \quad (27)$$

and

$$H_k \phi \equiv \left(-\frac{\partial^2}{\partial r^2} + k(k - \sigma_3) \frac{1}{r^2} + \sigma_1 \frac{1}{r} \right) \phi = \tilde{\varepsilon} \phi \quad (28)$$

correspondingly, provided we set $\tilde{\lambda} = \omega \lambda$.

The electromagnetic field whose potential is defined by relations (5) and (26) has a transparent physical meaning. Namely, it is a superposition of the electric field $\mathbf{E} = (E_1, E_2, E_3)$ whose components are

$$E_1 = \omega \frac{x_1}{x^2}, \quad E_2 = \omega \frac{x_2}{x^2}, \quad E_3 = 0 \quad (29)$$

and the magnetic field $\mathbf{B} = (B_1, B_2, B_3)$ with

$$B_1 = -\omega \frac{x_2}{x^2}, \quad B_2 = \omega \frac{x_1}{x^2}, \quad B_3 = 0. \quad (30)$$

Such an electric field can be identified as the field of a charged infinite filament coinciding with the third coordinate axis. Let us designate the charge density of this filament by ρ then the coupling constant ω should be equal to 2ρ . On the other hand, the magnetic field \mathbf{B} is nothing but the field of a straight line constant current J directed along the third coordinate axis provided the coupling constant ω be equal to $2J$. Of course the related charge density and current should be equal between themselves, i.e.,

$$j = \rho = \omega/2. \quad (31)$$

Let us show that equation (27) is exactly solvable and find its solutions. The simplest way to prove integrability of (27) is to make the unitary transformation

$$\chi \rightarrow \chi' = U\chi, \quad \varepsilon' - \left(\frac{p^2}{2M} + \tilde{\lambda} \frac{\sigma_\alpha x_\alpha}{x^2} \right) \rightarrow U \left(\varepsilon' - \left(\frac{p^2}{2M} + \tilde{\lambda} \frac{\sigma_\alpha x_\alpha}{x^2} \right) \right) U^\dagger \quad (32)$$

where $U = \frac{1}{\sqrt{2}}(1 - i\sigma_3)$. As a result we reduce (27) to the following form:

$$\varepsilon' \chi = \left(\frac{p^2}{2M} - 2\tilde{\lambda} \frac{S_1 x_2 - S_2 x_1}{x^2} \right) \chi \quad (33)$$

where $S_1 = \frac{1}{2}\sigma_1$ and $S_2 = \frac{1}{2}\sigma_2$ are fermionic spin matrices and in accordance with eqs.(3), (26) and (31) $\tilde{\lambda} = \lambda\omega = 2g\mu_0 j$.

For a fixed M and up to the value of the coupling constant equation (33) coincides with the Schrödinger equation for a neutral particle minimally interacting with the field generated by an infinite thin current filament (in our case the standard coupling constant is multiplied by factor 2). This equation was studied in numerous papers starting with [4] and continuing with [5]–[7] and many others. It has the following nice properties:

- equation (33) admits a hidden dynamical symmetry with respect to group $SO(3)$ for negative eigenvalues $\tilde{\varepsilon}$, group $SO(1, 2)$ for $\tilde{\varepsilon}$ positive and group $E(2)$ for $\tilde{\varepsilon} = 0$ [4];
- it possesses a hidden supersymmetry [5];
- using any of the above mentioned properties the equation can be integrated in closed form [4]–[6].

Since equation (27) is unitarily equivalent to eq.(33) it has the same properties. In particular, eigenvalues ε' are the same in both equations (27) and (33).

5 Relativistic and quasi relativistic energy levels

In the next section we will present exact solutions of equation (27) for coupled states and define the related eigenvalues ε' . In fact these eigenvalues are well known, and using directly results of paper [4] (or of the papers [5]-[8]) we can immediately write ε' in the following form:

$$\varepsilon' = -\frac{2\tilde{\lambda}^2 M}{N^2} \quad (34)$$

where N is a positive natural number.

Eigenvalues (34) are degenerated since they do not depend on eigenvalues k of the angular momentum operator J_{12} . The degeneration factor is equal to $2k + 1$, and the quantum number N can be represented as

$$N = 2(n + k) + 1 \quad (35)$$

where n is a natural number [4], [6].

Using (34) we already can find energy levels for the initial relativistic problem. Indeed, since $P_0 = p_0 + p_3$ and $P_3 = \frac{1}{2}(p_0 - p_3)$, it is possible to write analogous relations for eigenvalues E of p_0 , κ of p_3 and ε, M :

$$\varepsilon = E + \kappa, \quad 2M = E - \kappa. \quad (36)$$

Then, using definitions (27) and (36) for ε' , E and M we find from (34) the relativistic energy spectrum:

$$E = \frac{m}{1 + \frac{\tilde{\lambda}^2}{N^2}} \left(\frac{\tilde{\lambda}^2 \tilde{\kappa}}{N^2} + \sqrt{1 + \tilde{\kappa}^2 + \frac{\tilde{\lambda}^2}{N^2}} \right) \quad (37)$$

where $\tilde{\kappa} = \kappa/m$ and $\tilde{\lambda} = 2\mu_0 gj$.

We see that in spite of the fact that the neutron motion along the third coordinate axis is free, the third component of momentum κ makes a rather non-trivial contribution into the values of energy levels (37). In accordance with (37) $E > \kappa$, and so the condition $M \neq 0$ in fact is satisfied.

The most simple expression for energy levels corresponds to the particular value $\kappa = 0$:

$$E = \frac{m}{\sqrt{1 + \frac{\tilde{\lambda}^2}{N^2}}} \quad (38)$$

which for small $\tilde{\lambda}$ becomes

$$E = m \left(1 - \frac{\tilde{\lambda}^2}{2N^2} \right) + \dots = m - \frac{2m(g\mu_0 j)^2}{N^2} + \dots \quad (39)$$

Up to the rest energy term m the approximate energy levels (39) are exactly the same as in the non-relativistic PS problem [4], [6]–[8]. In particular both the approximate and exact levels given by equations (37), (38) and (39) are degenerated with respect to eigenvalues k of the third component of angular momentum which is a constant of motion for the considered system. Like in [4] this degeneration is caused by a hidden dynamical symmetry of the system.

Both $\tilde{\lambda}$ and $\tilde{\kappa}$ can be treated as small parameters. Expanding E (39) in power series of $\tilde{\lambda}$ and $\tilde{\kappa}$ we obtain the quasirelativistic approximation for the energy levels:

$$E \approx m + \frac{\kappa^2}{2m} - \frac{\kappa^4}{8m^3} - \frac{m\tilde{\lambda}_\kappa^2}{2N^2} \quad (40)$$

where

$$\tilde{\lambda}_\kappa = (1 - \tilde{\kappa})\tilde{\lambda}. \quad (41)$$

The first tree terms in (40) represent respectively the rest energy, the kinetic energy of the free motion along the third coordinate axis and the relativistic correction to this energy. The last (dynamical) term in (40) is quite similar to the corresponding non-relativistic term (compare with (39)), but includes the corrected coupling constant $\tilde{\lambda}_\kappa$ instead of $\tilde{\lambda}$.

Notice that since the electromagnetic field defined by relations (5) and (6) has no components in x_3 direction, the motion of the particle described by equation (10) in this direction is free. This free motion can be quantized by imposing the periodic boundary condition. Then

$$\kappa = \frac{2\pi\tilde{N}}{L}, \quad \tilde{N} = 0, \pm 1, \pm 2, \dots \quad (42)$$

and energy levels (37)–(40) be labeled by the pairs of quantum numbers N and \tilde{N} .

6 Exact solutions for bound states

To find the solutions of eq.(27) we use the fact that the Hamiltonian

$$H_k = \left(-\frac{\partial^2}{\partial r^2} + k(k - \sigma_3) \frac{1}{r^2} + \sigma_1 \frac{1}{r} \right) \quad (43)$$

can be factorized as

$$H_k = a_k^+ a_k + C_k \quad (44)$$

where

$$a_k = \frac{\partial}{\partial r} + W_k, \quad a_k^+ = -\frac{\partial}{\partial r} + W_k, \quad C_k = -\frac{1}{(2k + 1)^2}$$

and W is a *matrix superpotential*

$$W_k = \frac{1}{2r}\sigma_3 - \frac{1}{2k+1}\sigma_1 - \frac{(k+\frac{1}{2})}{r}. \quad (45)$$

It can be verified by direct calculation that

$$H_k^+ = a_k a_k^+ + C_k = -\frac{\partial^2}{\partial r^2} + (k+1)(k+1-\sigma_3)\frac{1}{r^2} + \tilde{\lambda}\sigma_1\frac{1}{r}$$

i.e., the superpartner Hamiltonian H_k^+ for H_k is equal to H_{k+1} . Thus the eigenvalue problem (24) possesses a supersymmetry with shape invariance and so it can be solved using the standard technique of the supersymmetric quantum mechanics [18]. We will not reproduce the related routine calculations whose details can be found in [6], [7] but restrict ourselves to the presentation of the solutions of eq.(27).

The ground state solutions $\phi(0, k; r) = \text{column } (\phi_1(0, k; r), \phi_2(0, k; r))$ are square integrable and normalizable solutions of equation $a_k\phi(0, k; r) = 0$. They can be expressed in the following form:

$$\phi_1(0, k; r) = r^{k+1}K_1(\frac{r}{2k+1}), \quad \phi_2(0, k; r) = -r^{k+1}K_0(\frac{r}{2k+1}) \quad (46)$$

where K_0 and K_1 are modified Bessel functions. The corresponding eigenvalue $\tilde{\varepsilon}_k$ in (27) and (24) is equal to $-\frac{1}{(2k+1)^2}$.

Solutions corresponding to the first excited state, i.e., to $n = 1$ are $\phi(1, k; r) = a_k^+ \phi(0, k+1)$, or, being written componentwise

$$\begin{aligned} \phi_1(1, k; r) &= -\left(\frac{\partial}{\partial r} + \frac{k}{r}\right)\phi_1(0, k+1; r) - \frac{1}{(2k+1)}\phi_2(0, k+1; r) \\ &= \frac{4(k+1)}{(2k+1)(2k+3)}r^{k+2}K_0(\frac{r}{2k+3}) - (2k+1)r^{k+1}K_1(\frac{r}{2k+3}), \\ \phi_2(1, k; r) &= -\left(\frac{\partial}{\partial r} + \frac{k+1}{r}\right)\phi_2(0, k+1; r) - \frac{1}{(2k+1)}\phi_1(0, k+1; r) \\ &= (2k+3)r^{k+1}K_0(\frac{r}{2k+3}) - \frac{4(k+1)}{(2k+1)(2k+3)}r^{k+2}K_1(\frac{r}{2k+3}). \end{aligned} \quad (47)$$

The corresponding eigenvalue $\tilde{\varepsilon}_k$ is equal to $-\frac{1}{(2(k+1)+1)^2} = -\frac{1}{(2k+3)^2}$.

Finally, solutions which correspond to an arbitrary value of the quantum number $n > 0$ can be represented as

$$\phi(n, k; r) = a_k^+ a_{k+1}^+ \cdots a_{k+n-1}^+ \phi(0, j+n; r), \quad n = 1, 2, \dots$$

which gives rise to the recurrence relations

$$\begin{aligned} \phi_1(n, k; r) &= -\frac{\partial}{\partial r}\phi_1(n-1, k+n-1; r) - \frac{k+n-1}{r}\phi_1(n-1, k+n-1; r) \\ &\quad + \frac{1}{2(k+n)-1}\phi_2(n, k+n-1; r), \\ \phi_2(n, k; r) &= -\frac{\partial}{\partial r}\phi_2(n, k+n-1; r) - \frac{k+n-1}{r}\phi_2(n, k+n-1; r) \\ &\quad + \frac{1}{2(k+n)-1}\phi_1(n, k+n-1; r). \end{aligned} \quad (48)$$

The related eigenvalue $\tilde{\varepsilon}_k$ is given by relations (34) and (35).

It is now possible to present exact solutions of the initial Dirac-Pauli equation defined by relations (1), (5) and (26). In accordance with the above such solutions are labeled by the main quantum number N which can be expressed by equation (35), and by eigenvalues κ and k of the third component of momenta and total orbital momentum. Using equations (12), (13), (15), (21), (23) and (46)–(48) we find these solutions in the following form:

$$\psi_{n,\kappa,k} = \exp(-i(Ex_0 - \kappa x_3)) \frac{1}{\sqrt{2\pi Lr}} \begin{pmatrix} \exp(i(k - \frac{1}{2})\theta)\eta_1(n, k; r) \\ \exp(i(k + \frac{1}{2})\theta)\eta_2(n, k; r) \\ \exp(i(k - \frac{1}{2})\theta)\phi_1(n, k; r) \\ \exp(i(k + \frac{1}{2})\theta)\phi_2(n, k; r) \end{pmatrix}. \quad (49)$$

Here $r = (E - \kappa)\tilde{\lambda}\sqrt{x_1^2 + x_2^2}$, E and κ are given by equations (37) and (42), $\phi_1(n, k; r)$ and $\phi_2(n, k; r)$ are functions defined by recurrence relations (46), (48),

$$\begin{aligned} \eta_1(n, k; r) &= \tilde{\lambda} \left(\frac{\partial}{\partial r} + \frac{k}{r} \right) \phi_2(n, k; r) + \frac{m}{E - \kappa} \phi_1(n, k; r), \\ \eta_2(n, k; r) &= \tilde{\lambda} \left(\frac{\partial}{\partial r} - \frac{k}{r} \right) \phi_1(n, k; r) + \frac{m}{E - \kappa} \phi_2(n, k; r). \end{aligned} \quad (50)$$

Solutions (49) are normalizable and tend to zero with $r \rightarrow 0$. They are defined for non-negative eigenvalues k of the total angular momentum while solutions for k negative can be obtained acting on (49) by the reflection operator $\hat{Q} = i\gamma_0\gamma_2\gamma_3R_1$ where R is the reflection of the first spatial variable, i.e., $R_1\psi(x_0, x_1, x_2, x_3) = \psi(x_0, -x_1, x_2, x_3)$ and $R_1\psi(x_0, r, \theta, x_3) = \psi(x_0, , r, -\theta, x_3)$. Operator \hat{Q} transforms solutions into solutions and changes the sign of k . It is reduced to operator Q (20) when acting on the third and fourth components of bispinor (49).

Let us present components of solutions (49) for $n = 0$ and $n = 1$ in explicit form. If $n = 0$ then the related functions $\phi_1(n, k; r) = \phi_1(0, k; r)$ and $\phi_2(n, k; r) = \phi_2(0, k; r)$ are given by equation (46) while $\eta_1(0, k; r)$ and $\eta_2(0, k; r)$ have the following form:

$$\begin{aligned} \eta_1(0, k; r) &= \left(\frac{\tilde{\lambda}}{2k+1} + \frac{m}{E - \kappa} \right) r^{k+1} K_1 \left(\frac{r}{2k+1} \right) - \tilde{\lambda}(2k+1)r^k K_0 \left(\frac{r}{2k+1} \right), \\ \eta_2(0, k; r) &= - \left(\frac{\tilde{\lambda}}{2k+1} + \frac{m}{E - \kappa} \right) r^{k+1} K_0 \left(\frac{r}{2k+1} \right). \end{aligned}$$

If $n = 1$ then the corresponding functions $\phi_1(1, k; r)$ and $\phi_2(1, k; r)$ are given by equations (47), and

$$\begin{aligned} \eta_1(1, k; r) &= \left(\tilde{\lambda}(2k+1)(2k+3)r^k + \left(\frac{\tilde{\lambda}}{2k+3} + \frac{m}{E - \kappa} \right) \frac{4(k+1)}{(2k+3)(2k+1)} r^{k+2} \right) K_0 \left(\frac{r}{2k+3} \right) \\ &\quad - \left(\tilde{\lambda} \frac{6k+7}{2k+3} + (2k+1) \frac{m}{E - \kappa} \right) r^{k+1} K_1 \left(\frac{r}{2k+3} \right), \\ \eta_2(1, k; r) &= \left(\tilde{\lambda} \frac{3k+4}{2k+1} + \frac{m}{E - \kappa} (2k+3) \right) r^{k+1} K_0 \left(\frac{r}{2k+3} \right) \\ &\quad - \left(\frac{\tilde{\lambda}}{2} + \frac{m}{E - \kappa} \right) \frac{4(k+1)}{(2k+3)(2k+1)} r^{k+2} K_1 \left(\frac{r}{2k+3} \right). \end{aligned}$$

7 Discussion

In Sections 2 and 3 we study a rather extended class of Dirac-Pauli systems which can be effectively reduced to a set of Schrödinger-Pauli equations. But the main scope of the present paper was to find an integrable relativistic formulation of the non-relativistic PS problem [4]. This goal cannot be achieved by a straightforward relativization of the PS problem, i.e., keeping the external field as being generated by a filament current j and changing the Schrödinger-Pauli equation by the Dirac-Pauli one. The simple reason is that proceeding in this way one obtains a model which is non integrable.

In this paper we succeed in obtaining an integrable relativistic analogue of the PS model. To this end we add an extra external field generated by an infinite charged line. Moreover the charge density ρ is related to the current j by equation (31) or, in SI units:

$$\rho = j/c^2 \quad (51)$$

where c denotes the speed of light. In this way we come to the relativistic model defined by relations (1), (5) and (26). This model is indeed integrable, its exact solutions for bound states are given in Section 6 and the corresponding energy levels are discussed in Section 5.

In accordance with (51) the required charge density is very small and tends to zero in the non-relativistic approximation when $c \rightarrow \infty$. Thus we claim that the non-relativistic limit of the model defined by equations (1), (5) and (26) is exactly the PS model. This property can be directly proved using the Foldy-Wouthuysen transformation [19]. We shall not present such a proof here but restrict ourselves to comparing the energy levels predicted by our relativistic and non-relativistic PS models.

In accordance with (38) in the non-relativistic approximation these levels simple coincide (of course up to the rest energy term), which confirms our supposition. We also find corrections for the PS levels in the quasi-relativistic approximation (40). Albeit the motion along the third Cartesian coordinate is free, the third momentum component κ makes indeed a contribution into the effective coupling constant $\tilde{\lambda}_\kappa$. The origin of this contribution is the anomalous interaction of moving neutral Dirac fermion with the magnetic field caused by the charged line. In the rest frame the motion along this line is effectively changed by the current which flows in the line in the opposite direction, which is in perfect accordance with equation (41).

The contribution of κ into the effective coupling constant $\tilde{\lambda}_\kappa$ (41) is small, namely, in SI units it is proportional to the inverse speed of light. However, it affects the energy levels (40) much more than the relativistic correction to the kinetic energy $-\kappa^4/8m^3$ which is proportional to the squared inverse speed of light.

Considering the exact solutions presented in Section 6 it is easy to show that the probability density corresponding to wave functions (49) admits the following

approximate representation:

$$\overline{\psi}_{n,\kappa,k} \gamma_0 \psi_{n,\kappa,k} = \sqrt{\frac{2}{\pi Lx}} (\phi_1^2 + \phi_2^2) + \dots$$

where the dots denote terms of order $1/c^2$. In other words, up to the terms proportional to the inverse squared speed of light, the probability density looks as in the non-relativistic PS problem. Thus many of conclusions concerning the probability distributions for the non-relativistic PS problem (such as the long lifetime for bound states with sufficiently large k) made in [8] and [6] can be generalized to the relativistic case.

In conclusion we have presented a new exactly solvable problem for the Dirac-Pauli equation describing a neutral particle which interacts anomalously with a rather particular external field given by equations (29)-(31) having however a clear physical meaning. This type of anomalous interaction is the key to expand solutions of the problem via solutions of the (1+2)-dimensional Levi-Leblond equation invariant with respect to Galilei group. Moreover, the considered problem possesses a hidden symmetry and supersymmetry which cause the $(2n+1)$ -fold degeneration of the energy levels given by equations (37) and (35).

A natural question arises whether the considered relativistic problem with its symmetries is unique or there are other problems which can be effectively solved using reduction $SO(1,3) \rightarrow HG(1,2)$. In Sections 2 and 3 we study a certain class of such problems which can be effectively reduced to radial equation (24) which is exactly solvable when $e = 0$ and $\varphi = \omega \log(x)$. We believe that there are other exactly solvable equations (24) and at least two of them can be immediately written down if we set $\varphi = \alpha/x$ and consider the alternative cases $e = 0$ and $e \neq 0$. The related Dirac-Pauli equations (1), (5), (6) can be solved explicitly. We plan to study these and probably other integrable models in future.

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